Taking the Mystery out of Spiral Curves

by Jim Coan

Part 1

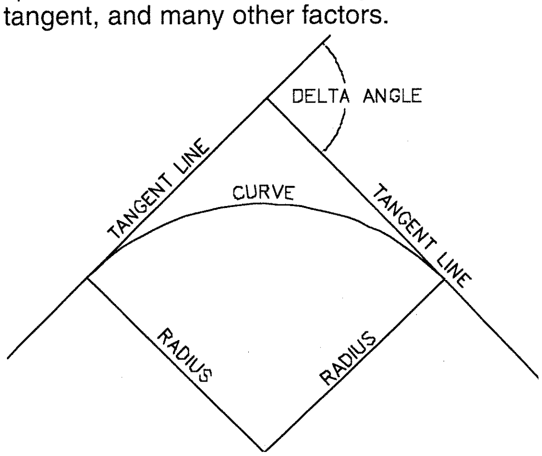
Quite often I am contacted by Land Surveyors and asked to help them with the calculation or interpretation of a spiral curve. These curves are usually on some historic record with a minimum of information. In order to figure out what was done the spiral curve needs to be reconstructed (re-calculated).

"For many surveyors, the last time they had to calculate a spiral curve was in college,

Spiral curves are no harder than many other things a surveyor calculates. In fact, they are much easier than a lot of things we are asked to do on a daily basis. The reason some surveyors have such a hard time with spiral curves is because we very seldom deal with them. For many surveyors, the last time they had to calculate a spiral curve was in college, if at all. I feel if a surveyor understands how a spiral is constructed and has the proper formulas, they will have little trouble with this type of curve.

First, it must be stated that there are two common types of spiral curves, Arc Spirals, and Chord Spirals. Arc Spirals are used in road work with the design of highways. Many states don't use spiral curves at all, instead opting to use radial curves with longer radiuses. Other states use to design highways with spirals but have stopped the practice. There are a few states that still use spirals. Chord spiral curves are typically found in rail road work. For the purpose of this paper arc spiral curves will be discussed.

A spiral curve is just the modification of a simple horizontal curve. First an appropriate curve is placed between two tangent lines. This curve is designed with respect to the design speed of the road, the delta angle between the tangent, and many other factors.



Once the radius (or in many cases the degree of curve), the design speed, and the geometry of the cross section (how many lanes) are known, the length of the spiral portion of the curve can be found in highway design tables.

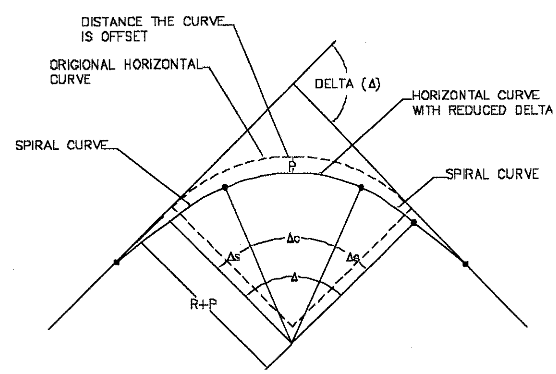
The simple curve now changes from one curve to a curve complex with three parts. They are the spiral in, the central curve, and the spiral out. In many cases the spiral portion of the curve is also the super elevation transition of the curve complex.

First, the radial curve is reduced from its original delta angle to a delta angle that is smaller. Then the radial curve is offset towards the radius point, but the radius stays the same as in the original curve. The difference between the original delta angle and the reduced delta angle is two times the delta angle of the spiral section. That is

$$\Delta c = \Delta - 2\Delta s$$

Where Δ= the delta angle of the overall curve complex, Δc = the delta angle of the reduced curve, and Δs = the delta angle of the spiral. The Δs is calculated by the following formula.

$$\Delta s = \frac{90°}{\pi}\,\frac{Ls}{R} \text{ OR } \Delta = \frac{LsD}{200}$$



When this is done, the curve is transformed form a simple horizontal curve to a spiral curve. There are now several new parts to the curve. The point where the curve complex starts is called the Tangent to Spiral (TS). The point where the spiral in meets the horizontal portion of the curve is called the Spiral to Curve (SC). Where the horizontal curve meets the spiral out, the point is called Curve to Spiral (CS), and where the spiral out meets the tangent line is called the Spiral to Tangent (ST)

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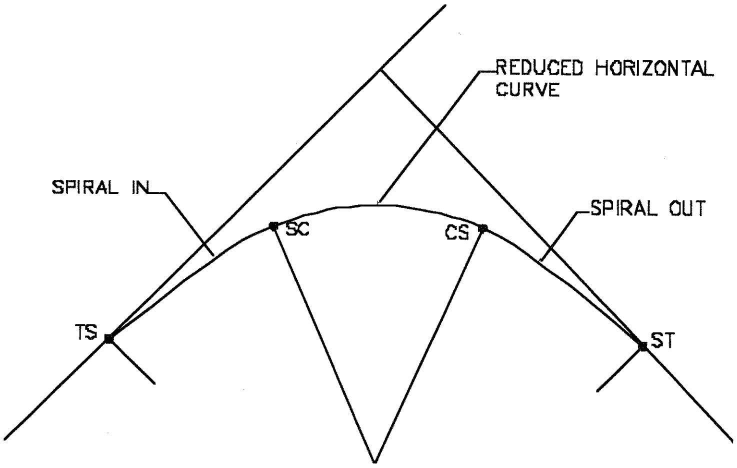
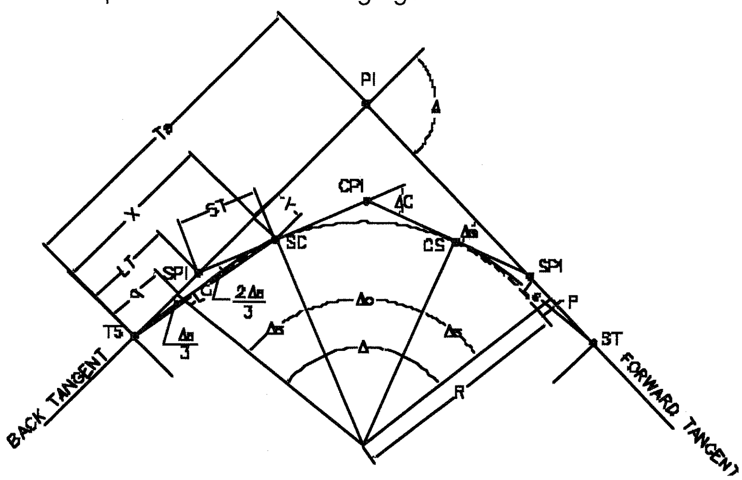


The geometry of a spiral curve is complex. Most parts of the curve can be calculated once it is understood where they are on the spiral. See the following figure



Ts is the Spiral Tangent. This is the distance from the PI of the overall curve complex to where the spiral begins (TS). The formula to calculate this distance is:

$$Ts = (R + P)Tan\frac{\Delta}{2} + q$$

X is a distance, measured along the tangent from the TS to a point at right angles to the SC. The formula to calculate this distance is:

$$X = Ls\left[1 - \frac{\Delta^2 s}{5(2!)} + \frac{\Delta^4 s}{9(4!)} - \frac{\Delta^6 s}{13(6!)} + ...\right]$$

This can become a very complicated formula in a hurry, however, I have found that if only the first two terms inside the brackets are used the answer will be accurate to the 0.01 feet. When this is done the formula will look like this.

$$X = Ls\left[1 - \frac{\Delta^2 s}{10}\right]$$

This formula is much easier to use. **<u>It must also be noted that when the above two formulas are used the Δs must be in radians</u>**

Another way to find a value for "X" is to use an approximation formula. If the length of the spiral is treated like a straight line then the "X" and the "Y" can be calculated by using the formulas of a right triangle. This will only be an approximation but might be precise enough for a particular job.

For example, given a length of spiral (*Ls*) of 200.00 feet, and a delta spiral of 03°49'11" (0.0666666 radians based on a radius of 1500.00 feet)

The "X" value, using the series formula will be 199.91 feet

The "X" value, using the right angle approximation will be 199.95 feet

Y is the right angle distance from the tangent to the SC. The formula to calculate this distance is:

$$Y = Ls\left[\frac{\Delta s}{3} - \frac{\Delta^3 s}{7(3!)} + \frac{\Delta^5 s}{11(5!)} - \frac{\Delta^7 s}{15(7!)} + ...\right]$$

As with the "X" value, the formula for the "Y" value can be simplified to the two terms inside the brackets. The above formula then becomes

$$Y = Ls\left(\frac{\Delta s}{3} - \frac{\Delta^3 s}{42}\right)$$

Again, the Δs must be in radians!

Just like the "X" value you can also use a right angle approximation.

Example: Using the same parameters as the example for the "X" value the "Y" value using the series formula is 4.44 feet. Using the right angle approximation the "Y" value is also 4.44 feet.

LT is the long tangent (spiral). It is the distance from the TS to the spiral point of intersection (SPI) as measured along the tangent line.

The formula to find the LT is:

$$LT = \frac{LC \bullet Sin\frac{2\Delta s}{3}}{Sin(180° - \Delta s)}$$

In this formula the Δs does not need to be changed to radians.

ST is the short tangent (spiral). It is the straight line distance from the SPI to the spiral to curve (SC). The formula is:

$$ST = \frac{LC \bullet Sin\frac{2\Delta s}{3}}{Sin(180° - \Delta s)}$$

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Taking the Mystery *(continued)*

Ls is the arc length of the spiral, it is a function of the radius (or degree of curve), the design speed of the road, and how many lanes the road is designed for. The Ls can be determined from highway design tables such as the one shown below.

TABLE 10.4(a) *Spiral Curve Lengths and Superelevation Rates: (a) Superelevation (e) Maximum of 0.06, Typical for Northern Climate.*

(a)

	V = 30			V = 40			V = 50			V = 60			V = 70			V = 80		
		L, ft			L, ft			L, ft			L, ft			L, ft			L, ft	
D	e	2 Lane	4 Lane	e	2 Lane	4 Lane	e	2 Lane	4 Lane	e	2 Lane	4 Lane	e	2 Lane	4 Lane	e	2 Lane	4 Lane
0°15'	NC	0	0	NC	0	0	NC	0	0	NC	0	0	NC	0	0	RC	250	250
0°30'	NC	0	0	NC	0	0	NC	0	0	RC	200	200	RC	200	200	.023	250	250
0°45'	NC	0	0	NC	0	0	RC	150	150	.021	200	200	.026	200	200	.033	250	250
1°00'	NC	0	0	RC	150	150	.020	150	150	.027	200	200	.033	200	200	.041	250	250
1°30'	RC	100	100	.020	150	150	.028	150	150	.036	200	200	.044	200	200	.053	250	300
2°00'	RC	100	100	.026	150	150	.035	150	150	.044	200	200	.052	200	250	.059	250	300
2°30'	.020	100	100	.031	150	150	.040	150	150	.050	200	200	.057	200	300	.060	250	300
3°00'	.023	100	100	.035	150	150	.044	150	200	.054	200	250	.060	200	300	D_{max} = 2°30'		
3°30'	.026	100	100	.038	150	150	.048	150	200	.057	200	250	D_{max} = 3°00'					
4°00'	.029	100	100	.041	150	150	.051	150	200	.059	200	250						
5°00'	.034	100	100	.046	150	150	.056	150	200	.060	200	250						
6°00'	.038	100	100	.050	150	200	.059	150	250	D_{max} = 4°30'								
7°00'	.041	100	150	.054	150	200	.060	150	250									
8°00'	.043	100	150	.056	150	200	D_{max} = 7°00'											
9°00'	.046	100	150	.058	150	200												
10°00'	.048	100	150	.059	150	200												
11°00'	.050	100	150	.060	150	200												
12°00'	.052	100	150	D_{max} = 11°00'														
13°00'	.053	100	150															
14°00'	.055	100	150															
16°00'	.058	100	200															
18°00'	.059	150	200															
20°00'	.060	150	200															
21°00'	.060	150	200															
D_{max} = 21°00'																		

In the above table the value of "*D*" in the left hand column stands for degree of curve the *e* is for the maximum rate of superelevation around a curve, and the *L* is for the length of the spiral (Ls) in feet for a given design speed.

LC is the chord length of the spiral. This is a straight line distance from the TS to the SC. Given the value of "X" and "Y" the LC can be found by using right triangle formulas

$$LC = \sqrt{X^2 + Y^2}$$

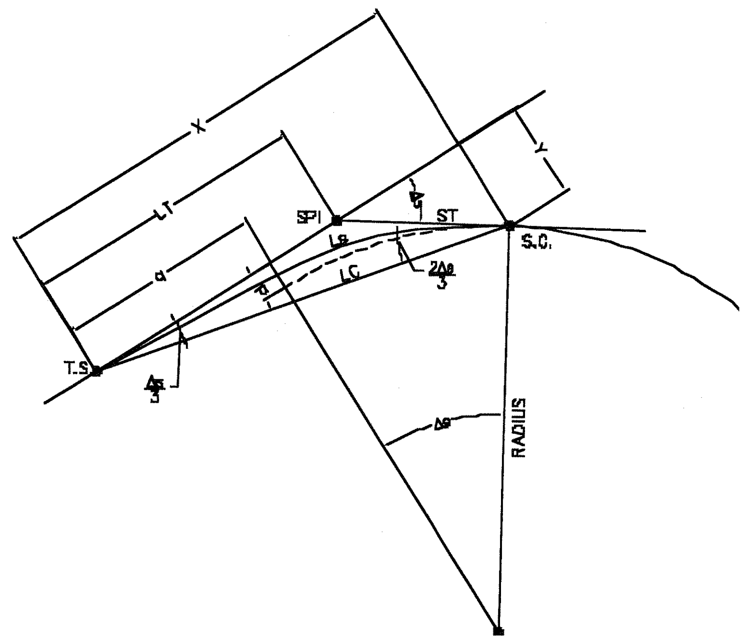
q is the distance along the tangent to a point at right angles to the ghost PC (just less than Ls÷2). The formula for finding "q" is:

$$q = X - R \cdot Sin\ \Delta s$$

P is the distance the curve is offset. The formula to find "P" is:

$$P = Y - R\ (1 - Cos\ \Delta s)$$

At this point it should be stated that $\frac{\Delta s}{3}$ and $\frac{2\Delta s}{3}$ are approximate values, but very good approximate values. For most spiral curves these values will be exact to the nearest one second of a degree. When Δs is as large as 21° (which is very rare) the correction to $\frac{\Delta s}{3}$ is approximately 30".



In part two of our discussion we will talk about the parts of a spiral curve.

LOOK FOR PART 2 IN THE COMING WINTER ISSUE.

Taking The Mystery Out Of Spiral Curves—Part 2

(see the Fall Issue for Part 1)

By Jim Coan, Renton Technical College

PARTS OF A SPIRAL CURVE

An example of calculating all parts of a spiral curve is as follows.

Given:
A four lane road with a design speed of 70mph; a degree of curve of 03º00’00”; a Delta Angle of 60º00’00”. With the above information a radius will be calculated at 1909.86 feet. From table 10.4a (above) the length of the spiral curve will be 300.00 feet.

With this information all parts of the spiral curve can be calculated.

First, from the above information Ds, and Dc can be calculated. To calculate Ds we use the formula $\Delta_S = \frac{90^\circ}{\pi} \bullet \frac{L_S}{R}$, for our curve $\Delta_S = \frac{90^\circ}{\pi} \bullet \frac{300}{1,909.86} = 04^\circ 30'00"$ and

$\Delta_C = \Delta - 2\Delta_S$, again, for our curve $\Delta_C = 60^\circ 00'00" - 2(04^\circ 30'00") = 51^\circ 00'00"$

Now that Ds has been calculates, “X” and “Y” can be found. First we will calculate “X” using the formula:

$$X = L_S\left(1 - \frac{\Delta_S^{\ 2}}{10}\right), \text{ for our curve } X = 300.00'\left(1 - \frac{(0.078539816)^2}{10}\right) = 299.815'$$

*Remember in this formula as well as the formula to find “Y” Ds must be in radians.

To find “Y” we will use the formula:

$$Y = L_S\left(\frac{\Delta_S}{3} - \frac{\Delta_S^{\ 3}}{42}\right), \text{ for our curve } Y = 300.00'\left(\frac{0.078539816}{3} - \frac{(0.078539816)^3}{42}\right) = 7.851'$$

Next we will calculate the length of the spiral chord (LC) using the formula:

$LC = \sqrt{X^2 + Y^2}$, for our curve $LC = \sqrt{299.815^2 + 7.851^2} = 299.918'$

With the information we have calculated so far we can now calculate the long tangent (LT), and the short tangent (ST). To find the LT we will use the formula:

$$LT = \frac{LC \bullet Sin\frac{2\Delta_S}{3}}{Sin(180^\circ - \Delta_S)}, \text{ for our curve } LT = \frac{299.918 \bullet Sin\frac{2(04^\circ 30'00")}{3}}{Sin(180^\circ - 04^\circ 30'00")} = 200.060'$$

To find the ST we will use the formula:

$$ST = \frac{LC \bullet Sin\frac{\Delta_S}{3}}{Sin(180^\circ - \Delta_S)}, \text{ for our curve } ST = \frac{299.918' \bullet Sin\frac{04^\circ 30'00"}{3}}{Sin(180^\circ - 04^\circ 30'00")} = 100.064'$$

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Next, "P" and "q" need to be calculated. The formula to find "P" is:

$P = Y - R(1 - Cos\Delta_s)$, for our curve $P = 7.851 - 1,909.86(1 - Cos04^\circ30'00") = 1.964'$

The formula to find "q" is:

$q = X - R \bullet Sin\Delta_S$, for our curve $q = 299.815 - 1,909.86(Sin04^\circ30'00") = 149.969'$

Finally, after all calculations have been done the spiral tangent can be found by using the formula:

$T_S = (R + P)Tan\frac{\Delta}{2} + q$, for our curve $T_S = (1,909.86 + 1.964)Tan\frac{60^\circ00'00"}{2} + 149.969 = 1,253.761'$

The final results are:

D = 60°00'00"	X = 299.815'
Dc = 51°00'00"	Y = 7.851'
Ds = 04°30'00"	LT = 200.060'
Dc = 03°00'00"	ST = 100.064'
R = 1,909.860'	P = 1.964'
Ls = 300.000'	q = 149.969'
LC = 299.918'	Ts = 1,253.761'

Now that the entire spiral has been calculated the next step is to make the necessary calculations to lay out the spiral. In order to get the necessary information two parts must be calculated, the deflection angle and the chord distance for intermediate points on the spiral.

First, the chord distance needs to be determined. There are many ways to accomplish this but the best way is to use the arc distance as a chord distance. This will create an error because chord distances are always shorter than arc distances, but for the precision of 0.01 feet the error is very small if at all and it is very complicated to calculate a true chord distance. The following table illustrates the error of the chord to spiral over a 300 foot spiral curve.

Station	Arc Distance	Chord Distance
TS = 0+00	0.00'	0.00'
0+50	50.00'	50.00'
1+00	100.00'	100.00'
1+50	150.00'	150.00'
2+00	200.00'	199.99'
2+50	250.00'	249.97'
SC=3+00	300.00'	299.92'

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This is based on a 300.00 foot spiral with a Δs of 04°30'00". Because the LC has already been calculated, the largest error of 0.08 feet at 3+00 (SC) can be dismissed leaving the largest practical error of 0.03 feet. For layout this should be accurate enough. If more precision is needed there several programs available such as TDS or Land Development Desktop that do a very good job of calculating spiral chord distances.

Next, the deflection angle is calculated. In order to do this we will use the formula

$\alpha_S = \left(\frac{l}{L_S}\right)^2 \bullet \frac{\Delta_S}{3}$ where "l" is the length on the spiral to the point being staked. *Ls* is the length of the spiral (300.00feet), and Δs is the delta spiral (04°30'00")

Example:

Station	Distance	Deflection Angle
TS = 0+00	0.00'	00°00'00"
0+50	50.00'	00°02'30"
1+00	100.00'	00°10'00"
1+50	150.00'	00°22'30"
2+00	200.00'	00°40'00"
2+50	250.00'	01°02'30"
SC = 3+00	300.00'	01°30'00"

Again, these deflection angles are not perfect because $\frac{\Delta_S}{3}$ is not perfect but it is more than adequate for the surveying precision needed to layout a spiral curve.

The next step, now that all of the calculations are done, is to layout the curve complex. This will consist of laying out the spiral in, the center curve and the spiral out.

If you asked ten different surveyors how to lay out a spiral curve complex you would probably get ten deferent answers, and this surveyor is no different. I will explain my preferred procedure. There are many other ways and the best way is the one the surveyor responsible for the work is comfortable with.

First, if possible the P.I. of the entire curve complex should be occupied. Once this is done the back tangent can be sited and the TC as well as the SPI of the spiral in can be set. While still at the P.I. the delta angle can be turned as a deflection angle and the SPI and ST can be set along the ahead tangent.

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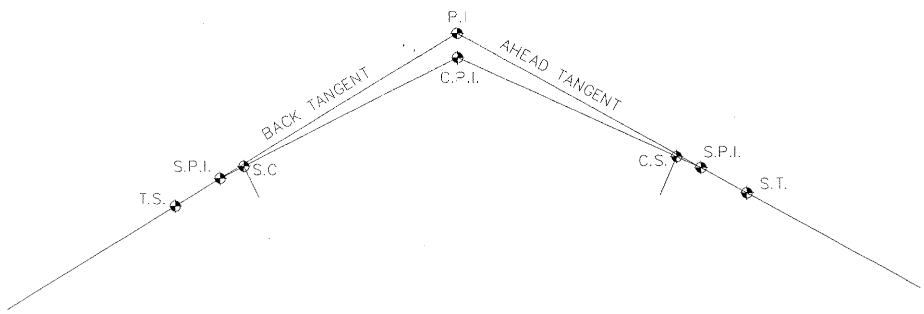
Once these points are set the SPI of the back tangent can be occupied, the instrument can be sited along the back tangent, the spiral delta angle can be turned as a deflection angle, and the SC as well as the CPI can be set. Next occupy the CPI, backsite the SPI that was just occupied, turn the delta angle of the interior curve as a deflection angle and set the CS. At this point the surveyor should also be looking at the SPI that was set for the ahead tangent when the PI was occupied (here is your check).



This is a "best case" scenario and will not always fit the conditions in the field. Regardless of the conditions, if the control points can be put in first the surveyor will keep better control over the spiral curve complex.

Once the control is in, the TS can be occupied, the tangent line can be sited as the backsite, the deflection angles turned, the distances measured and the spiral in can be surveyed.

The center curve can be surveyed as any horizontal curve would be staked out. This procedure is well known to surveyors and will not be discussed further.

The spiral out can be stakes backwards (or back stations), that is, the instrument can be set up on the ST the ahead tangent can be used as a backsite and the angles that are already calculated can be turned as angles left, or subtracted from 360° and turned as angles right.

There is another way that a spiral can be surveyed. This can be accomplished by occupying the CS and staking out the spiral to the ST. This involves dividing the spiral into even segments. For example our 300 foot spiral can be divided into 6 even 50 foot stations. Each part of the spiral will be given a number beginning with 0 at the ST and going to 6 at the CS. These numbers will be referred to as Chord Point Numbers (CPN)

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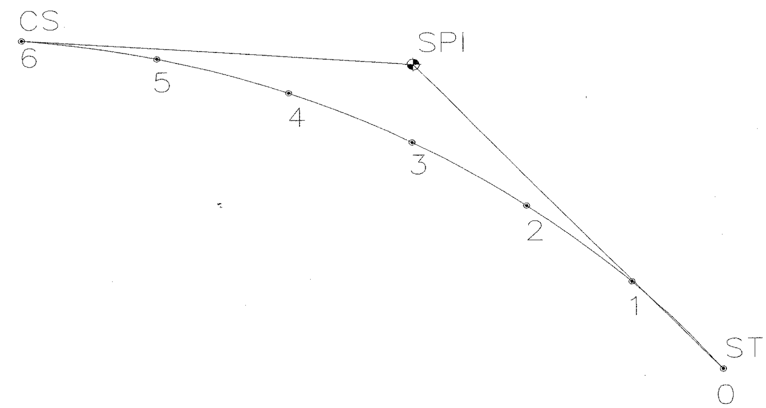

The deflection angles can be calculated by using the formula;

$$\alpha = C\left[\frac{\left(\frac{\Delta_S}{3}\right)}{n^2}\right]$$

Where "C" is a constant calculated by the formula $C = (n_f + 2n_t)(n_t - n_f)$. In this equation n_f is the CPN of the point being staked, and $n_t = n =$ the number where the instrument is set up. Another way to find "C" is by the following table

"C" VALUES CONSTANT FOR SPIRAL CURVES												
DEFLECTION TO CPN	INSTRUMENT AT CHORD POINT NUMBER (n)											DEFLECTION TO CPN
	0	1	2	3	4	5	6	7	8	9	10	
0	0	2	8	18	32	50	72	98	128	162	200	0
1	1	0	5	14	27	44	65	90	119	152	189	1
2	4	-4	0	8	20	36	56	80	108	140	176	2
3	9	-10	-7	0	11	26	45	68	95	126	161	3
4	16	-18	-16	-10	0	14	32	54	80	110	144	4
5	25	-28	-27	-22	-13	0	17	38	63	92	125	5
6	36	-40	-40	-36	-28	-16	0	20	44	72	104	6
7	49	-54	-55	-52	-45	-34	-19	0	23	50	81	7
8	64	-70	-72	-70	-64	-54	-40	-22	0	26	56	8
9	81	-88	-91	-90	-85	-76	-63	-46	-25	0	29	9
10	100	-108	-112	-112	-108	-100	-88	-72	-52	-28	0	10

(Continued on page 31)

Spiral Curve *(continued)*

In the above table, if the instrument is at CPN 6 and the deflection angle for CPN 4 is being calculated, the "C" value for the formula will be 32. When the "C" value is acquired either by calculation or by the table the absolute value should be used. This procedure can work on the spiral in as well as the spiral out. The trick is dividing the spiral into even sections.

Example:

STATION	CPN	C	DEF. ANGLE	DISTANCE
CS = 10+00	6	0	0	0
10+50	5	17	00°42'36"	50.00'
11+00	4	32	01°20'00"	100.00'
11+50	3	45	01°52'30"	150.00'
12+00	2	56	02°20'00"	200.00'
12+50	1	65	02°42'30"	250.00'
ST = 13+00	0	72	03°00'00"	300.00'

In the above example the deflection angle for the last station (13+00) is the same as $\frac{2\Delta_S}{3}$ again, this is your check. The distances are the station distance (arc distance) as previously explained.

As previously stated, this is but one common way to calculate a spiral curve complex. Hopefully this discussion has taken the mystery out of spiral curves, their parts, and the calculations involved. Granted, there are numerous steps in the process, but no equations are too complex for a surveyor. Once all parts are calculated the surveyor can use whatever they need to accomplish their job. If more information is required, there are several books on route surveying or highway design manuals readily available.