GMSPD25

Workshop on Geometric Mechanics, Structure Preserving Discretizations, and Discrete Differential Geometry

Book of Abstracts

SUNY Polytechnic Institute
Utica, NY, USA
July 21-25, 2025

In the last decades, different fields emerged to discretize and simulate physical problems in a structure-preserving, efficient and stable manner (**DEC, FEEC, DDG**). On the other hand, geometric mechanics (**GM**) became a powerful framework in continuum mechanics and generalized materials, such as plasticity and defect theory and, nowadays, also for (manufactured) metamaterials. The advances of both groups rely on the same techniques from modern differential geometry and algebraic topology, with numerical analysis bridging the gap.

Cartan's exterior calculus expresses the derivative operators (div, grad, curl) in terms of the more fundamental wedge product, exterior derivative, and Hodge star operators and separates metric information from topology. This unifies the classical integral theorems (Green's, Stokes', Gauss's), with de Rham cohomology being the link between differential forms and topology. The translation of the equations of elasticity to exterior calculus came somewhat later compared to electromagnetism, fluids, and general relativity; it first had to be worked out that the stress tensor is a vector-valued two form.

The discretization of scalar-valued differential forms on a manifold is now well-understood. However, the discretization of differential forms with values in a vector-bundle with a connection is notoriously difficult and requires advanced concepts from functional analysis, modern differential geometry, algebraic topology, and numerical mathematics.

GM for Cosserat materials and dislocations, where the manifold describing the location of the material is different from the manifold for the additional degrees of freedom, involves vector-bundle valued differential forms.

From **DEC** we know that pairing values on chains (cochains) with chains is the discrete analogue of integration of a continuous scalar-valued differential form over a domain. Taking the discrete exterior derivative as the adjoint of the boundary operator on chains then leads to a structure-preserving discretization of the Stokes' theorem or the de Rham cohomology. Other DEC operators are a robust discrete Hodge star and a discrete wedge product. DEC has been successfully applied to the Navier-Stokes equations.

FEEC used cohomology and Hodge theory to derive a unified framework to build a discrete de Rham complex that is isomorphic to its continuous version. Milestones were the derivation and analysis of stable mixed finite element discretizations of the Hodge-Laplacian eigenvalue problem and Maxwell's equations. The work of so-called twisted complexes enabled the description of more involved physical problems, such as Cosserat elasticity and linearized Riemann-Cartan geometry in this framework.

The field of **DDG** aims to develop a framework for the discrete counterpart of differential geometry on smooth manifolds and their convergence. Focus has been on the crucial case of polyhedral surfaces consisting of triangles and quadrilaterals approximating a smooth surface embedded in R3. Several discrete curvature operators and other differential geometry objects, such as connections, have been defined on the approximated manifolds, and convergence of these objects under mesh refinements has been investigated.

Acknowledgements

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We aim to advance the understanding of the cross-section of Geometric Mechanics, Numerical Mathematics, and Discrete Differential Geometry. Knowledge is created, shared, applied, and evolves, "It needs to be nourished to flourish". By reaching a diverse audience, the workshop contributes to a broader infrastructure for knowledge creation and preservation in the fields of GM and SPDs and to strengthening the national scientific community.

Program

Monday July 21

Welcome by Dean Dr. Babak Elahi

9 am — **Andrea Dziubek** — SUNY Polytechnic Institute, NY

A gentle introduction to vector valued differential forms in geometric mechanics and their structure preserving discretizations

10:00 — **Coffee break**

10:30 — **Michael Karow** — TU Berlin, Germany

On compatibility conditions in Riemann-Cartan geometry (GM)

11:30 — **Lunch break**

1 pm — **Nicholas Andrzejkiewicz** — SUNY Polytechnic Institute, NY

Reviewing Ehresmannian geometry in electromagnetism and elasticity (GM)

2:00 — **Michael Neunteufel** — Portland State University, OR

FEM meets Discrete Differential Geometry: Extrinsic and intrinsic curvature approximation (DDG)

3:00 — **Coffee break**

3:30 — **Jack McKee** — University of Hawai'i at Mānoa, Hawai'i

Defining curvature for Regge metrics (DDG)

4:30

Wednesday, July 23

Welcome by Provost Dr. Andrew Russell

9 am — **Kaibo Hu** — University of Edinburgh and Maxwell Institute, UK

Finite element form valued forms (FEEC, online)

10:00 — **Coffee break**

10:30 — **Yasha Berchenko-Kogan** — Florida Inst. of Technology, FL

Two approaches for discretizing spaces of tensors with specified interelement continuity conditions (FEEC, online)

11:30 — **Lunch break**

1 pm — **Evan Gawlik** — Santa Clara University, CA

Double forms: Decomposition and discretization (FEEC)

2:00 — **Martin Licht** — Swiss Federal Tech. Inst. of Lausanne, Switzerland

New estimates for potential operators in vector calculus and exterior calculus (FEEC)

3:00 — **Coffee break**

3:30 — **Anil Hirani** — University of Illinois, IL

A tutorial on exterior covariant derivatives of double forms (DEC)

4:30 — **Chris Eldred** — Sandia National Lab., University of New Mexico, NM

A structure-preserving Lie-Poisson scheme for parabolically regularized compressible flow (DEC)

Abstracts

A gentle introduction to vector valued differential forms in geometric mechanics and their structure preserving discretizations

Andrea Dziubek

SUNY Polytechnic Institute, Utica, NY

Structure preserving discretizations (SPD) for partial differential equations have been around for about two decades now. They have proven to accurately, efficiently and stable solve e.g. the Maxwell equations on nontrivial domains, the Navier-Stokes equation on curved surfaces, the equations for thin elastic shells, and the Cosserat media equations.

Key in the development of geometric mechanics (GM) was the realization that the description of generalized (Cosserat) materials and the theory of dislocations are related. Micropolar (Cosserat) material is described as a field of infinitesimal rigid bodies whose orientations are determined by a field of orthonormal frames, in other words, as a principal fiber bundle. And dislocation theory corresponds to an incompatible Cosserat continuum.

There has been limited interaction between the GM communities and the various SPD communities. Some work has been done to extend these discretizations to general fibered manifolds. However, this requires advanced concepts from modern differential geometry – beyond tensor calculus on Riemannian manifolds which has long been used to describe classical mechanics.

This talk will give an introduction to both GM and SPD, with special focus on general fibered manifolds from different perspectives.

On compatibility conditions in Riemann-Cartan geometry

Michael Karow

TU Berlin, Germany

We review basic compatibility conditions as the Poincare Lemma and the Frobenius Theorem. Their relationship to the curvature and the torsion tensor are discussed using a non-standard notation.

Reviewing Ehresmannian Geometry in Electromagnetism and Elasticity

Nicholas Andrzejkiewicz

SUNY Polytechnic Institute, Utica, NY

Ehresmannian geometry generalizes the notions of connection and curvature to structure groups G other than SO(3), SO(n), or SO(1,3). It includes Riemannian geometry as a subset by ignoring metric information and starting with the Levi-Civita connection.

This framework includes examples such as electromagnetism G=U(1), symplectic geometry/quantization G= U(1), elasticity G=SO(3), the classical strong force G=SU(3), and the classical weak force G=SU(2). Even in the context of Riemannian geometry it more clearly contains and explains the complicated transformation laws of the gamma/Christoffel symbols in Riemannian geometry and relativity.

Ehresmannian geometry is developed in terms of Lie-algebra-valued differential forms, not on space or spacetime, but on the G-principal bundles over space or spacetime, generalizing the notions of frames and coordinates. The choice involved in taking a cross section of this higher dimensional space is why Christoffel symbols transform as they do.

References

[1] Arnold, V. I.: Mathematical Methods of Classical Mechanics, Translated from Russian by K. Vogtmann and A. Weinstein, Grad. Texts in Math., 60, Springer, 1978.

[2] Kobayashi, Shoshichi and Nomizu, Katsumi: Foundations of Differential Geometry. Vol I, John Wiley & Sons, 1963.

[3] Sternberg, Shlomo: Curvature in Mathematics and Physics, Dover, 2012.

[4] Bleecker, D.: Gauge Theory and Variational Principles, Dover, 2005.

FEM meets Discrete Differential Geometry: Extrinsic & intrinsic curvature approximation

Jay Gopalakrishnan[1], Michael Neunteufel[1], Joachim Schöberl[2], Max Wardetzky[3]

[1] Portland State University
[2] TU Wien
[3] University of Göttingen

The curvature of surfaces and Riemannian manifolds is essential in several fields such as nonlinear shell analysis, general relativity, and geometric flows. Discrete differential geometry (DDG) seeks to approximate curvature quantities on discretized surfaces and manifolds. Incorporating DDG into a finite element method (FEM) framework provides robust tools for analysis and development of algorithms. Distributional finite elements are crucial here. However, due to their weak regularity, (nonlinear) differential operators have to be interpreted in a sense of distributions.

In this talk, we present how the DDG algorithms of dihedral angles for extrinsic curvature and angle defects for intrinsic curvature approximation can be incorporated into FEM. To this end, we approximate the surface using Lagrange finite elements and the Riemannian metric tensor by symmetric, tangential-tangential continuous Regge elements [1, 2]. We discuss convergence results by utilizing an integral representation of the distributional curvatures involving covariant differential operators [3]. We present the application of distributional extrinsic curvature in nonlinear shell models. Numerical examples are demonstrated with the finite element library NGSolve (www.ngsolve.org) and NGSDiffGeo, an add-on package for differential geometry support.

References

[1] Christiansen S.H.: On the linearization of Regge calculus. Numerische Mathematik 119, 4 (2011).

[2] Li L.: Regge Finite Elements with Applications in Solid Mechanics and Relativity. Ph.D. thesis, University of Minnesota (2018).

[3] Gopalakrishnan J., Neunteufel M., Schöberl J., Wardetzky M.: Generalizing Riemann curvature to Regge elements. arXiv, 2311.01603 (2024).

Defining Curvature for Regge Metrics

Jack McKee[1], Evan Gawlik[2]

[1] University of Hawai'i
[2] Santa Clara University

Various weak forms of the Riemann curvature tensor have been proposed and evaluated, especially for the special class of metrics called Regge metrics, which are piecewise smooth and have tangential-tangential continuity. Usually these are inspired by mimicry of certain properties of the smooth curvature tensor. I will present a new definition of the curvature functional for Regge metrics, inspired by direct weakening of the Cartan structure equations. By picking a special type of orthonormal frame, it is possible to derive an equation for this functional that is equivalent to existing equations for the distributional Riemann curvature, but has the attractive property of being expressible using only integration of differential forms on manifolds.

Finite element form-valued forms

Kaibo Hu[1], Ting Lin[2]

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[2] School of Mathematics, Peking University

Classical finite element methods, such as those by Lagrange, Nédélec, Raviart–Thomas, and Brezzi-Douglas-Marini, fit within de Rham complexes and can be interpreted as discrete differential forms. These finite element differential forms encode discrete topology and have become standard practice for solving vector-valued problems. Their structures also find broad applications in discrete topology, including topological data analysis and the Hodge Laplacian on graphs.

In this work, we focus on tensors with applications in continuum mechanics, differential geometry, and general relativity. First, we investigate the algebraic and differential structures of tensor fields. We show that tensor fields with natural symmetries fit within Bernstein-Gelfand-Gelfand (BGG) complexes and twisted de Rham complexes, and we discuss the correspondence between these complexes, generalized continua, and Riemann-Cartan geometry. Second, we construct finite elements for form-valued forms (double forms). Special cases include classical finite element differential forms, distributional finite elements, Christiansen's finite element interpretation for Regge calculus in quantum and numerical gravity (discrete metric and curvature), the TDNNS/HHJ element for elasticity, the MCS element for Stokes equations, and various new spaces.

The talk is based on several collaborative works [1]-[4].

References

[1] Arnold, D. N., & Hu, K. (2021). Complexes from complexes. Foundations of Computational Mathematics, 21(6), 1739-1774.

[2] Čap, A., & Hu, K. (2024). BGG sequences with weak regularity and applications. Foundations of Computational Mathematics, 24(4), 1145-1184.

[3] Hu, K., Lin, T., & Zhang, Q. (2025). Distributional Hessian and divdiv complexes on triangulation and cohomology. SIAM Journal on Applied Algebra and Geometry, 9(1), 108-153.

[4] Hu, K., & Lin, T. (2025). Finite element form-valued forms: Construction. arXiv preprint arXiv:2503.03243.

Two approaches for discretizing spaces of tensors with specified interelement continuity conditions

Yakov Berchenko-Kogan[1], Evan Gawlik[2]

[1] Florida Institute of Technology
[2] Santa Clara University

This talk will discuss two projects: finite element spaces of double forms and blow-up finite elements. While the approaches are quite different, both projects ultimately address the same question: Given a space of tensors, how does one construct finite element spaces with desired interelement continuity conditions?

As we know from finite element exterior calculus, vector fields with tangential continuity are best thought of as one-forms, and vector fields with normal continuity are best thought of as $(n-1)$-forms, with interelement continuity enforced via restrictions to the interface. A key advantage of the differential form perspective is that the spaces and the continuity conditions are *metric-independent*, so the finite element spaces work just as well for tangent vector fields on polyhedral surfaces as they do for vector fields on triangulations of the plane, for example. Analogously, for matrix fields, we can have tangential–tangential continuity, tangential–normal continuity, or normal–normal continuity, and these finite element spaces are best thought of as spaces of double forms, that is, sections of $\Lambda^p \otimes \Lambda^q$. We construct (when possible) finite element spaces for all values of p and q, all possible symmetries imposed on the tensor, and all polynomial coefficient degrees r, and we expect our approach to generalize to other metric-independent spaces of tensors as well.

In contrast, vector fields with full continuity are *metric-dependent* and behave somewhat differently. With a Euclidean metric, the situation is simple: we can use Lagrange elements. However, with a non-Euclidean metric, using Lagrange elements leads to a problem: if the angle defect is nonzero, the continuity conditions are overconstrained at vertices, forcing the vector field to be zero there. We develop new finite element spaces called *blow-up finite elements* with rational function coefficients that are flexible enough to resolve this issue. With vector-valued (or, more generally, tensor-valued) blow-up finite elements, we can impose whatever interface continuity conditions we like, even in the presence of a nonzero angle defect. Additionally, blow-up finite elements can be extended to a complex of blow-up Whitney forms, so we can express vector-valued (or tensor-valued) differential forms in this framework as well.

Double forms: Decomposition and discretization

Yakov Berchenko-Kogan[1], Evan Gawlik[2], Anil Hirani[3]

[1] Florida Institute of Technology
[2] Santa Clara University
[3] University of Illinois Urbana-Champaign

Tensors with symmetries arise in a variety of situations that are modeled by partial differential equations. Of particular prevalence are $(p+q)$-tensors that alternate in their first p first arguments and alternate in their last q arguments. These tensors are called double forms or (p, q)-forms.

This talk will discuss the space $\Lambda^{p,q}$ of (p, q)-forms, its algebraic structure, and its discretization with finite elements. The talk will expand upon some of the content presented by Yakov Berchenko-Kogan and will include some joint work with Anil Hirani. It will highlight two important algebraic tools: a canonical decomposition of $\Lambda^{p,q}$, and a natural map s from $\Lambda^{p,q}$ to $\Lambda^{p+1,q-1}$ that antisymmetrizes the first $p+1$ arguments. I will discuss a few ways of deriving the decomposition, its interplay with the map s, and its role in the construction of finite element spaces for $\Lambda^{p,q}$. I will also discuss some algebraic aspects of the map s, including its behavior under composition and inversion.

New estimates for potential operators in vector calculus and exterior calculus

Theophile Chaumont-Frelet[1], Martin Werner Licht[2], Martin Vohralik[3]

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[2] EPFL, Switzerland
[3] INRIA Paris, France

We discuss Poincare–Friedrichs inequalities in the context of vector calculus and exterior calculus. This includes the Poincare inequality for the gradient operator and its generalizations for the curl operator and the exterior derivative. Estimating the optimal constants in these inequalities reduces to estimating operator norms of the associated potential operators. We present several special cases and obtain upper bounds for convex Lipschitz domains using results by Guerini and Savo, along with new estimates for the regularized Poincare and Bogovskii operators.

More generally, we examine Poincare–Friedrichs constants over local finite element patches within triangulated domains, using the notion of shellability from the theory of polytopal complexes. Finally, we extend these results to general triangulated domains, deriving reliable and computable bounds for the Poincare–Friedrichs constants of differential operators. Diagram chasing within a Cech–de Rham complex reduces this to a merely finite-dimensional problem that is easily assembled from the geometric setting. Numerical experiments support the theoretical results. Part of this work is joint with Theophile Chaumont-Frelet and Martin Vohralik.

Tutorial Introduction to Exterior Covariant Derivatives of Double Forms

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[2] Santa Clara University

Exterior covariant derivatives of double forms are families of operators $d_\nabla^{j,k} : \Omega^{j,k}(M) \to \Omega^{j+1,k}(M)$. These are a special case of the covariant exterior derivatives $d_{\nabla^E}^j$ that act on the spaces $\Omega^j(M;E)$ of vector bundle valued forms for a vector bundle E over M with connection ∇^E. The goal of the tutorial is to familiarize the audience with the algebraic rules that $d_\nabla^{j,k}$ and $d_{\nabla^E}^j$ satisfy without worrying about analysis stuff or technical nonsense about vector bundles. This will be done using several small example computations done entirely without coordinates. We will see how $d_\nabla \circ d_\nabla$ gives curvature, how d_∇ simplifies in $\mathbb{R}^n$, how torsion shows up when d_∇ interacts with the Bianchi sums (diagonal operators of BGG construction), and what happens to the Hessian sequence in this general setting [1].

References

[1] Evan Gawlik and Anil N. Hirani, *Sequences from sequences, sans coordinates*, In preparation.

A structure-preserving Lie-Poisson scheme for parabolically regularized compressible flow

Christopher Eldred

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It is well known that the compressible Euler equations can be written in Lie-Poisson Hamiltonian form in terms of mass density, momentum density and (thermodynamic) entropy density. Doing so exposes several fundamental features, including conservation laws such as total mass and total energy. Using structure-preserving numerics, these properties can be preserved at the discrete level. In this work, we have chosen to use a discrete exterior calculus (DEC) scheme in space and a discrete gradient (DG) scheme in time. Specifically, we use a structure-preserving, high-resolution, oscillation-limiting, bounds-preserving (SPHROL-BP) discretization of the Lie derivative. This provides excellent numerical solutions for the case of smooth flows, and facilitates preservation of the invariant domain for the Euler equations.

However, in the presence of solution discontinuities, the Lie-Poisson formulation is physically incorrect since it implies conservation of (thermodynamic) entropy across shocks. This is dealt with by introducing a nonlinear, solution and grid dependent parabolic regularization that parameterizes the entropy generation across shocks, and ensures positivity of density and a maximum principle for entropy. This regularization can be written as a type of metriplectic system that conserves total energy and generates (thermodynamic) entropy, which generalizes Hamiltonian dynamics and can be discretized using the same type of structure-preserving DEC-DG scheme. Combining these elements, we obtain a fully discrete scheme with exact (local) conservation of mass and energy, generation of thermodynamic entropy across shocks, and invariant-domain preservation.

Results will be demonstrated for a variety of 1D flow problems, including various Riemann problems. If time permits, there will be discussion of extending these ideas to the case of charged fluids such as MHD and Euler-Maxwell, where the underlying Lie-Poisson Hamiltonian formulation facilitates the design of schemes with charge conservation and other involution constraints.